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NON-AXISYMMETRIC LOADINGS OF A CRACKED CYLINDER

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Abstract—A general method is developed to determine the stress-intensity factor for a penny-shaped crack embedded in an elastic circular cylinder and deformed by non-axisymmetric normal stresses. The surface of the cylinder is assumed to be stress free. Numerical values of the stress-intensity factor are presented for a cylinder subjected to transverse bending.

INTRODUCTION

The problem of a concentric penny-shaped crack embedded in an infinite circular cylinder and subjected to axisymmetric normal loading has been considered by Collins[1], Sneddon and Tait[2] and Sneddon and Welch[3]. When the crack surfaces are subjected to in-plane shear and torsional stresses the formulas for the stress-intensity factors are given in the work of Kassir and Sih[4]. In addition, the last reference contains results applicable to other shapes of finite solids with buried penny-shaped cracks (e.g. thick plates, half-spaces and spheres).

The purpose of this investigation is to present a general procedure for solving a class of problems involving a penny-shaped crack embedded in a long circular cylinder and opened by nonaxisymmetric pressure distribution. The crack is assumed to be concentrically located in a mid-plane normal to the axis of the cylinder, and the material of the cylinder is idealized to be homogeneous, isotropic and linearly elastic. It is also assumed that the surface of the cylinder is free from stress (Other conditions can be dealt with in analogous manner).

Since the geometry of the cylinder is symmetric about the crack plane, it suffices to formulate the problem for a semi-infinite cylindrical region. At any point in this region, the displacement field consists of two parts: One part is associated with an unbounded solid containing a penny-shaped crack and the other part is associated with an uncracked cylinder. An integral transform solution appropriate to the first part has been developed by Smith *et al.*[5] (see also Keer[6]). In this investigation, a displacement solution of the non-axisymmetric field equations of elasticity is utilized to develop an integral transform solution appropriate to the cylindrical region. The two solutions are employed to reduce the problem to integral equations of Fredholm type. For the illustrative example of a cylinder subjected to transverse bending, the integral equations are solved numerically and the stress-intensity factors are computed for several values of the crack radius.

FORMULATION OF PROBLEM AND SOLUTION

Consider a penny-shaped crack of radius a embedded in a long circular cylinder of radius $b(b > a)$. It is assumed that the center of the crack is located on the axis of the cylinder and its plane is normal to that axis. Figure 1 shows the geometry of the problem where the position of a point is defined by cylindrical coordinates (r, θ, z). In this coordinate system, the crack occupies the region $z = 0^{\mp}$, $0 \leq \theta \leq 2\pi$, $0 \leq r \leq a$, the displacement components are denoted by u_r, u_θ, u_z and σ_r, σ_θ, σ_z, $\tau_{r\theta}$, τ_{rz} and $\tau_{\theta z}$ designate the corresponding stress components. The state of stress set up in the neighborhood of the crack is induced by identical distribution of

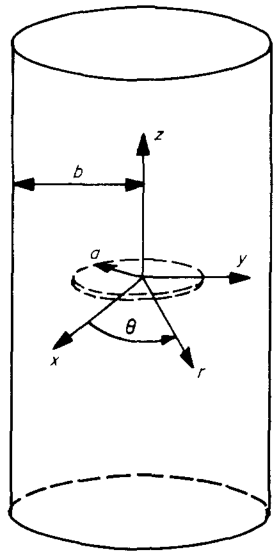

Fig. 1. An embedded crack in a circular cylinder.

normal pressure, $\sigma_z(r,\theta,0) = -p(r,\theta)$, applied to the two surfaces of the crack while the surface of the cylinder is assumed to be stress free.

Referring to the semi-infinite region $z \geqslant 0, 0 \leqslant r \leqslant b, 0 \leqslant \theta \leqslant 2\pi$, the boundary conditions can be expressed in the form

$$\tau_{rz} = \tau_{\theta z} = 0, \qquad z = 0, \qquad 0 \leqslant \theta \leqslant 2\pi, \qquad 0 \leqslant r \leqslant b, \tag{1}$$

$$\sigma_z = -p(r.\ \theta) = -\sum_{n=0}^{\infty} H_n(r)\cos(n\theta), \qquad z = 0,$$
$$0 \leqslant \theta \leqslant 2\pi, \qquad 0 \leqslant r \leqslant a, \tag{2}$$

$$u_z = 0, \qquad z = 0, \qquad 0 \leqslant \theta \leqslant 2\pi, \qquad a \leqslant r \leqslant b, \tag{3}$$

$$\sigma_r = \tau_{r\theta} = \tau_{rz} = 0, \qquad 0 \leqslant z < \infty, \qquad r = b, \qquad 0 \leqslant \theta \leqslant 2\pi, \tag{4}$$

where all stresses tend to zero as $z \to \infty$ and the Fourier coefficients, $H_n(r)$, in eqn (2) are given by

$$H_0(r) = \frac{1}{\pi}\int_0^{\pi} p(r,\theta)\,\mathrm{d}\theta$$
$$H_n(r) = \frac{2}{\pi}\int_0^{\pi} p(r,\theta)\cos(n\theta)\,\mathrm{d}\theta, \quad n = 1,2,\ldots \ . \tag{5}$$

The displacement field in an infinite solid with a penny-shaped crack subjected to normal load of the type in eqn (2) is given by [5, 6]

$$2\mu\, u_r = (1-2\nu)\frac{\partial g}{\partial r} + z\frac{\partial^2 g}{\partial r \partial z}, \tag{6a}$$

$$2\mu\, u_\theta = (1-2\nu)\frac{1}{r}\frac{\partial g}{\partial \theta} + \frac{z}{r}\frac{\partial^2 g}{\partial \theta \partial z}, \tag{6b}$$

$$2\mu\, u_z = -2(1-\nu)\frac{\partial g}{\partial z} + z\frac{\partial^2 g}{\partial z^2} \tag{6c}$$

in which $g(r, \theta, z)$ is a harmonic function given by

$$g = \sum_{n=0}^{\infty} \cos(n\theta) \int_0^{\infty} \frac{A_n(s)}{s} J_n(rs)\, e^{-sz}\, ds. \tag{7}$$

In eqns (6) and (7), μ stands for the shear modulus of the material, J_n is the usual Bessel function of order n and $A_n(s)$ is an unknown function to be determined from boundary conditions (2) and (3). The corresponding solution to the problem of a finite radius cylinder without a crack can be accomplished by first expressing the displacement solution of the field equations of elasticity in the following form (see, e.g. Green and Zerna[7])

$$2\mu\, u_r = 4(1-\nu)(f_1 \cos\theta + f_2 \sin\theta) - \frac{\partial}{\partial r}(r\cos\theta f_1 + r\sin\theta f_2) + \frac{\partial f_3}{\partial r} + \frac{2}{r}\frac{\partial f_4}{\partial \theta}. \tag{8a}$$

$$2\mu\, u_\theta = 4(1-\nu)(-f_1 \sin\theta + f_2\cos\theta) - \frac{\partial}{\partial \theta}(f_1\cos\theta + f_2\sin\theta) + \frac{1}{r}\frac{\partial f_3}{\partial \theta} - 2\frac{\partial f_4}{\partial r}. \tag{8b}$$

$$2\mu\, u_z = -\frac{\partial}{\partial z}(rf_1\cos\theta + rf_z \sin\theta) + \frac{\partial f_3}{\partial z}, \tag{8c}$$

Here, $f_n(r, \theta, z)$, $n = 1,2,3,4$, are space harmonic functions. Since the normal stress described in boundary condition (2) has an even distribution in the variable θ, it follows from eqns (8) that f_1 and f_3 must also be even in θ while f_2 and f_4 are odd in the same variable. Moreover, as indicated in boundary condition (1), the shearing stresses vanish at all points of the $z = 0$ plane. In view of these observations and the harmonic character of the functions involved, the integral expressions of the displacement potentials in eqns (8) appropriate to the class of problems posed by boundary conditions (1)–(4) are found to be as follows

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \sum_{n=0}^{\infty} \begin{bmatrix} \cos(n+1)\theta \\ \sin(n+1)\theta \end{bmatrix} \int_0^{\infty} B_n(s)\, I_{n+1}(rs)\cos(\mathrm{sz})\, ds, \tag{9a}$$

$$f_3 = \sum_{n=0}^{\infty} \cos(n\theta) \int_0^{\infty} C_n(s)\, I_n(rs)\cos(sz)\, ds, \tag{9b}$$

$$f_4 = \sum_{n=1}^{\infty} \sin(n\theta) \int_0^{\infty} D_n(s)\, I_n(rs)\cos(sz)\, ds, \tag{9c}$$

where $I_n(rs)$ is modified Bessel function of the first kind and order n. It is clear from the form of eqns (8) and (9) that $B_n(s)$, $C_n(s)$ and $D_n(s)$ are integral transforms appropriate to the cylindrical region under general loading conditions. They are determined from the conditions imposed on the surface of the cylinder.

The total displacement field is obtained by adding eqns (6) and (8) and using the results given in eqns (7) and (9)

$$2\mu\, u_r = \sum_{n=0}^{\infty} \cos(n\theta) \int_0^{\infty} \Big\{ (1 - 2\nu - sz)\, A_n(s)\, J_n'(rs)\, e^{-sz} + \Big([(4 - 4\nu + n) I_{n+1}(rs) - rs\, I_n(rs)]\, B_n(s) + sI_n'(rs)\, C_n(S) + \frac{2n}{r} I_n(rs) D_n(rs) \Big) \cos(sz) \Big\}\, ds. \tag{10a}$$

$$2\mu\, u_\theta = \sum_{n=1}^{\infty} \sin(n\theta) \int_0^\infty \left\{ -\frac{n}{r}(1-2\nu-sz)\frac{A_n(s)}{s} J_n(rs)\,e^{-sz} + \left[(4-4\nu+n)\, I_{n+1}(rs)\, B_n(s) - \frac{n}{r} I_n(rs)\, C_n(s) - 2sI_n'(rs)\, D_n(s)\right] \cos(sz) \right\} ds, \tag{10b}$$

$$2\mu\, u_z = \sum_{n=0}^{\infty} \cos(n\theta) \int_0^\infty \left\{ (2-2\nu+sz)\, A_n(s)\, J_n(rs)\, e^{-sz} + [rI_{n+1}(rs)\, B_n(s) - I_n(rs)\, C_n(s)]\, s \sin(sz) \right\} ds, \tag{10c}$$

where the prime denotes differentiation with respect to the argument i.e. $s\, J_n'(rs) = (\partial/\partial r)\, J_n(rs)$. In the same manner, the stresses at an arbitrary point in the cylinder are found

$$\sigma_r = \sum_{n=0}^{\infty} \cos(n\theta) \int_0^\infty \Big\{ [(1-2\nu-sz)\, J_n''(rs) - 2\nu J_n(rs)]\, sA_n(s)\, e^{-sz} + \Big\langle [2\nu\, I_n(rs) + 2(1-2\nu)\, I_{n+1}'(rs) - srI_{n+1}''(rs)]\, sB_n(s) + s^2 I_n''(rs)\, C_n(s) + \frac{2n}{r^2}[s\, rI_n'(rs) - I_n(rs)]\, D_n(s) \Big\rangle \cos(sz) \Big\}\, ds, \tag{11a}$$

$$\sigma_z = \sum_{n=0}^{\infty} \cos(n\theta) \int_0^\infty \{ -(1+sz)\, A_n(s)\, J_n(rs)\, e^{-sz} + \langle [2\nu\, I_n(rs) + rs\, I_{n+1}(rs)]\, B_n(s) - I_n(rs)\, s\, C_n(s)\rangle \cos(sz)\}\, s\, ds, \tag{11b}$$

$$\tau_{rz} = \sum_{n=0}^{\infty} \cos(n\theta) \int_0^\infty \Big\{ s\, z\, A_n(s)\, J_n'(rs)\, e^{-sz} + \Big\langle [rs\, I_n(rs) - (n+2-2\nu)\, I_{n+1}(rs)]\, B_n(s) - sI_n'(rs)\, C_n(s) - \frac{n}{r} I_n(rs)\, D_n(s) \Big\rangle \sin(sz) \Big\}\, s\,ds, \tag{11c}$$

$$\tau_{\theta z} = \sum_{n=1}^{\infty} \sin(n\theta) \int_0^\infty \left\{ -\frac{n}{r} z\, A_n(s)\, J_n(rs)\, e^{-sz} + \left[(4\nu-3-n) I_{n+1}(rs)\, B_n(s) + \frac{n}{r} I_n(rs)\, C_n(s) + sI_n'(rs)\, D_n(s)\right] \sin(sz) \right\} s\, ds, \tag{11d}$$

$$\tau_{r\theta} = \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{r} \int_0^\infty \Big\{ \frac{n}{r}(1-2\nu-sz)\, [J_n(rs) - s\, rJ_n'(rs)]\, \frac{A_n(s)}{s} e^{-sz} + \Big\langle [(n+3-4\nu) rs\, I_n(rs) - (n+1)\,(n+5-6\nu)\, I_{n+1}(rs)]\, B_n(s) - \tfrac{1}{2}[(n-1)\, I_{n-1}(rs) + (n+1)\, I_{n+1}(rs)]\, s\, C_n(s) + [(n-n^2-r^2s^2)\, I_n(rs) + 2\, rs\, I_{n+1}(rs)]\, \frac{D_n(s)}{r} \Big\rangle \cos(sz) \Big\}\, ds. \tag{11e}$$

The expressions in eqns (11c) and (11d) satisfy the boundary conditions in eqn (1) automatically, and eqns (10c) and (11b) when used in conjunction with eqns (2) and (3) yield a pair of dual integral equations

$$\int_0^\infty s\,A_n(s)\,J_n(rs)\,\mathrm{d}s + \int_0^\infty s\{-[2\nu I_n(rs) + rs\,I_{n+1}(rs)]\,B_n(s) + I_n(rs)\,s\,C_n(s)\}\,\mathrm{d}s = H_n(r), 0 \leq r \leq a, \tag{12a}$$

$$\int_0^\infty A_n(s)\,J_n(rs)\,\mathrm{d}s = 0, \qquad a < r \leq b, \tag{12b}$$

whose solution is (Sneddon [8])

$$A_n(s) = s^{1/2} \int_0^a \phi_n(t) J_{n+1/2}(st)\,\mathrm{d}t, \tag{13}$$

provided that $\lim_{t\to 0} [t^{n-1/2}\,\phi_n(t)] = 0$. Making use of the result

$$\int_0^t \frac{r^{n+1}\,I_n(ry)\,\mathrm{d}r}{\sqrt{(t^2 - r^2)}} = \left(\frac{\pi}{2y}\right)^{1/2} t^{n+1/2}\,I_{n+1/2}(yt), \tag{14}$$

the functions $\phi_n(t)$, $n = 0,1,2,\ldots$, are found to be governed by

$$\phi_n(t) + t\int_0^\infty \langle [(2n+1-2\nu) I_{n+1/2}(yt) - yt\,I_{n-1/2}(yt)]\,B_n(y) + y I_{n+1/2}(yt)\,C_n(y)\rangle\,y^{1/2}\,\mathrm{d}y = \left(\frac{2}{\pi}\right)^{1/2} t^{-n+1/2} \int_0^t \frac{r^{n+1} H_n(r)\,\mathrm{d}r}{\sqrt{(t^2 - r^2)}} \quad 0 < t < a. \tag{15}$$

The next step in the analysis is to reduce eqn (15) into a standard integral equation. For this purpose and with a view toward satisfying the boundary requirements on the surface of the cylinder, it is found convenient to make use of the transformations

$$B_n(y) = \frac{y^{1/2}}{\pi} \int_0^a \phi_n(s) b_n(s, y)\,\mathrm{d}s, \tag{16a}$$

$$C_n(y) = \frac{y^{-1/2}}{\pi} \int_0^a \phi_n(s)\,c_n(s, y)\,\mathrm{d}s, \tag{16b}$$

$$D_n(y) = \frac{b\,y^{1/2}}{\pi} \int_0^a \phi_n(s)\,\mathrm{d}_n(s, y)\,\mathrm{d}s. \tag{16c}$$

With the aid of relations (16a) and (16b), eqn (15) is reduced to the Fredholm integral equations

$$\phi_n(t) + \int_0^a \phi_n(s)\,L_n(s, t)\,\mathrm{d}s = \left(\frac{2}{\pi}\right)^{1/2} t^{-n+1/2} \int_0^t \frac{r^{n+1}\,H_n(r)\,\mathrm{d}r}{(t^2 - r^2)^{1/2}}, \quad n = 0,1,2,\ldots \tag{17}$$

in which the kernels, L_n, are of the form

$$L_n(\lambda, t) = \frac{1}{\pi} \int_0^\infty ts\langle [(2n+1-2\nu)\,I_{n+1/2}(ts) - ts\,I_{n-1/2}(ts)]\,b_n(s, \lambda) + I_{n+1/2}(ts)\,c_n(s, \lambda)\rangle\,\mathrm{d}s. \tag{18}$$

In eqn (18), the functions $b_n(s, \lambda)$ and $c_n(s, \lambda)$ are determined from the boundary conditions in eqn (4).

Upon inserting the appropriate expressions of the stress components from eqns (11) into eqn (4), applying the inverse Fourier sine and cosine transforms in the variable z, making use of the relations in eqns (13) and (16) and evaluating the resulting Hankel integrals (Erdelyi [9]), the remaining unknown functions b_n and c_n are determined as

$$b_n(s, \lambda) = \frac{\alpha_n(s, \lambda)}{\Delta_n(s)}, \tag{19a}$$

$$c_n(s, \lambda) = \frac{\beta_n(s, \lambda)}{\Delta_n(s)}, \tag{19b}$$

where the expressions for α_n, β_n and Δ_n are given in the Appendix. When the functions b_n and c_n are inserted in the integral in eqn (18), the kernel L_n is readily shown to be bounded at both upper and lower limits, and the Fredholm equation may be solved for $\phi_n(t)$ either by iteration (for small values of the ratio a/b) or by numerical schemes (Kantorovich and Krylov[10]).

STRESS-INTENSITY FACTOR

In the vicinity of the crack border, the stress component $\sigma_z(r, \theta, 0)$ may be written as

$$\sigma_z(r, \theta, 0) = \frac{k_1}{\sqrt{(2r_1)}} + 0(r_1^0) \tag{20}$$

where $k_1(\theta)$ is the stress-intensity factor and $r_1 = r - a$, $r_1 \to 0$. It follows from eqns (11a) and (13) that k_1 is given by

$$k_1 = \frac{(2/\pi)^{1/2}}{a} \sum_{n=0}^{\infty} \phi_n(a) \cos(n\theta), \tag{21}$$

For large values of b, the kernel $L_n(s, t)$ in eqn (18) vanishes and $\phi_n(t)$ can be evaluated directly from eqn (17). By letting $t \to a$ and inserting the result in eqn (21) the expression for k_1 which applies to a penny-shaped crack embedded in an infinite solid and subjected to general normal loading conditions is recovered[5].

Similarly, the crack-opening displacement is obtained from eqns (10c) and (13). The result is

$$u_z(r, \theta, 0) = \left(\frac{2}{\pi}\right)^{1/2} \left(\frac{1-\nu}{\mu}\right) \sum_{n=0}^{\infty} r^n \cos(n\theta) \int_r^{\alpha} \frac{t^{-n-1/2}\phi_n(t)\,\mathrm{d}t}{\sqrt{(t^2 - r^2)}}, \quad 0 \leq r \leq a. \tag{22}$$

TRANVERSE BENDING OF CYLINDER

The formulation will be illustrated by considering the problem of a penny-shaped crack in a cylindrical beam subjected to transverse bending. In this case, $p(r, \theta)$ consists of

$$p(r, 0) = p_0 + p_1 \frac{r}{a} \cos\theta, \tag{23}$$

where p_0 and p_1 are constants. It follows from eqn (5) that

$$H_0(r) = p_0, \qquad H_1(r) = (p_1/a)r. \tag{24}$$

The functions $\phi_0(t)$ and $\phi_1(t)$ are governed by the integral equations

$$\phi_0(t) + \int_0^a \phi_0(s)\, L_0(s, t)\, \mathrm{d}s = \left(\frac{2}{\pi}\right)^{1/2} p_0\, t^{3/2}, \tag{25a}$$

$$\phi_1(t)+\int_0^a \phi_1(s)\,L_1(s,t)\,\mathrm{d}s=\frac{2}{3}\left(\frac{2}{\pi}\right)\frac{p_1}{a}t^{5/2}, \tag{25b}$$

in which

$$L_0(\lambda,t)=\frac{1}{\pi^{3/2}}\int_0^\infty (2ts)^{1/2}\langle[(1-2\nu)\sinh(ts)-ts\cosh(ts)]$$
$$\times b_0(s,\lambda)+\sinh(ts)\,c_0(s,\lambda)\rangle\,\mathrm{d}s, \tag{26a}$$

$$L_1(\lambda,t)=\frac{1}{\pi^{3/2}}\int_0^\infty\left(\frac{2}{ts}\right)^{1/2}\langle[(3-2\nu-ts)ts\cosh(ts)-(3-2\nu)\sinh(ts)]$$
$$\times b_1(s,\lambda)+[ts\cosh(ts)-\sinh(ts)]c_1(s,\lambda)\rangle\,\mathrm{d}s, \tag{26b}$$

and $b_j(s,\lambda)$ and $c_j(s,\lambda)$, $j=0,1$, are obtained from the appendix.

The stress-intensity factor is expressed in the form

$$k_1=\frac{2\,a^{1/2}}{\pi}\left[p_0\psi_0+\frac{2}{3}p_1\,\psi_1\cos\theta\right], \tag{27}$$

where the numerical values of ψ_0 and ψ_1 are obtained from eqns (25) after use is made of the substitutions

$$\phi_0(a)=(2/\pi)^{1/2}\,p_0\,a^{3/2}\,\psi_0$$
$$\phi_1(a)=\frac{2}{3}(2/\pi)^{1/2}\,p_1\,a^{3/2}\,\psi_1 \tag{28}$$

Equation (25b) was solved numerically for several values of the ratio a/b, and the values of ψ_1, together with the corresponding values of ψ_0 given in Ref. [3], are shown in Table 1 for a material with Poisson's ratio $v=\frac{1}{4}$. It should also be mentioned that in order to prevent contact between crack surfaces in the compression region of the cylinder ($[\pi/2]<\theta<[3\pi/2]$), the quantities p_0 and p_1 must be chosen so that $p_0\psi_0\geq(2/3)\,p_1\psi_1$ for all values of a/b.

Table 1. Numerical values of ψ_0 and ψ_1 in eqn (27)

(a/b)	ψ_0	ψ_1
0.2	1.005	1.0003
0.3	1.013	1.0017
0.4	—	1.0096
0.5	1.072	1.0224
0.6	—	1.0431
0.7	1.259	1.1382
0.8	1.479	1.2120
0.9	2.002	1.8721

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APPENDIX

Expressions for the functions α_n, β_n and Δ_n ($n = 1,2,\ldots$) in eqns (19) are obtained by expanding the determinants:

$$\Delta_n = |C_{ij}|, \qquad i,j = 1,2,3$$

$$\alpha_n = \begin{vmatrix} R_{11} & C_{12} & C_{13} \\ R_{21} & C_{22} & C_{23} \\ R_{31} & C_{32} & C_{33} \end{vmatrix}$$

$$\beta_n = \begin{vmatrix} C_{11} & R_{11} & C_{13} \\ C_{21} & R_{21} & C_{23} \\ C_{31} & R_{31} & C_{33} \end{vmatrix}$$

where

$$C_{11} = x\,I_n(x) - (n+2-2\nu)\,I_{n+1}(x)$$

$$C_{12} = -\tfrac{1}{2}[I_{n-1}(x) + I_{n+1}(x)]$$

$$C_{13} = -n\,I_n(x)$$

$$C_{21} = (n+3-4\nu)xI_n(x) - (n+1)(n+5-6\nu)\,I_{n+1}(x)$$

$$C_{22} = -\tfrac{1}{2}[(n-1)\,I_{n-1}(x) + (n+1)\,I_{n+1}(x)]$$

$$C_{23} = (n - n^2 - x^2)I_n(x) + 2xI_{n+1}(x)$$

$$C_{31} = (3-2\nu)xI_n(x) - [(n+1)(n+4-4\nu) + x^2]\,I_{n+1}(x)$$

$$C_{32} = (x^2 + n^2 - n)\left(\frac{1}{x}\right) I_n(x) - I_{n+1}(x)$$

$$C_{33} = 2n\,[xI_{n+1}(x) + (n-1)\,I_n(x)]$$

$$R_{11} = [-2x\,K_n(x) + K_{n-1}(x) + (1-2n)K_{n+1}(x)]\,I_{n+1/2}(y) + [K_{n-1}(x) + K_{n+1}(x)]\,y\,I_{n-1/2}(y)$$

$$R_{21} = \frac{2n}{x}\{\langle(2\nu - n - 2)xK_{n-1}(x) + [(n+1)(2\nu - 2n - 1) - x^2] \times K_n(x)\rangle\,I_{n+1/2}(y) + [xK_{n-1}(x) + (n+1)K_n(x)]y\,I_{n-1/2}(y)\}$$

$$R_{31} = \langle 2\nu[(n+1)K_{n+1}(x) - (n-1)K_{n-1}(x)] - 2nxK_n(x) + (n-1-x^2)K_{n-1}(x) - [(n+1)(2n+1) + x^2]\,K_{n+1}(x)\rangle\,I_{n+1/2}(y) + [2x\,Kn(x) + (n+1)K_{n+1}(x) - (n-1)K_{n-1}(x)]\,y\,I_{n-1/2}(y).$$

In these equations, $K_n(x)$ stands for the modified Bessel function of the second kind of order n, $x = bs$ and $y = ts$.

While the corresponding expressions for case $n = 0$ reduce to

$$\Delta_0(x) = x^2\,I_0^2(x) - (2 - 2\nu + x^2)\,I_1^2(x)$$

$$\alpha_0(x,y) = 2I_{1/2}(y)[1 - x^2 I_0(x)K_0(x) - (2 - 2\nu + x^2)\,I_1(x)K_1(x)] + 2y\,I_{-1/2}(y)$$

$$\beta_0(x,y) = 2\,I_{1/2}(y)\{4\nu - 4 - x^2 + (3-2\nu)\,[x^2 I_0(x)\,K_0(x) + (2 - 2\nu + x^2)\,I_1(x)\,K_1(x)]\} + 2y\,I_{-1/2}(y) \times [-2 + 2\nu + x\,I_0(x)\,K_0(x) + (2 - 2\nu + x^2)\,I_1(x)K_1(x)].$$